

Reconstruction of penetrable obstacles in the anisotropic acoustic scattering

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Abstract

We develop reconstruction schemes to determine penetrable obstacles in a region of \mathbb{R}^2 or \mathbb{R}^3 and we consider anisotropic elliptic equations. This algorithm uses oscillating-decaying solutions to the equation. We apply the oscillating-decaying solutions and the Runge approximation property to the inverse problem of identifying an inclusion in an anisotropic elliptic differential equation.

Keywords: enclosure method, reconstruction, oscillating-decaying solutions, Runge approximation property, Meyers L^p estimates.

1 Introduction

The special type solutions for elliptic equations or systems play an essential role in inverse problems since the pioneer work of Caldéron. In [12], Sylvester and Uhlmann used complex geometric optics (CGO) solutions to solve the inverse boundary value problems for the conductivity equation. Based on CGO solutions, Ikehata proposed the so called enclosure method to reconstruct the inclusion obstacle, see [11]. There are many results in this reconstruction algorithm, in [3], they construct CGO-solutions with polynomial-type phase function for the Helmholtz equation $\Delta u + k^2 u = 0$ or elliptic system having the Laplacian as the principal part. In [7], he constructed a very special solution of a conductivity equation $\nabla \cdot (\gamma(x) \nabla u) = 0$ (called the oscillating-decaying solutions), the leading parts is also isotropic. However, when the medium is anisotropic, we need to consider more general elliptic equations, such as anisotropic scalar elliptic equations $\nabla \cdot (A^0(x) \nabla u) + k^2 u = 0$, where $A^0(x) = (a_{ij}^0(x))$, $a_{ij}^0(x) = a_{ji}^0(x)$ and assume the uniform ellipticity condition, that is, $\forall \xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$, $\lambda^0 |\xi|^2 \leq \sum_{i,j} a_{ij}^0(x) \xi_i \xi_j \leq \Lambda^0 |\xi|^2$. In this paper, we want to use the oscillating-decaying solutions in our reconstruction algorithm. We have some assumptions. First, we consider this problem in \mathbb{R}^3 and assume that D is an unknown obstacle such that $D \Subset \Omega \subset \mathbb{R}^3$ with an inhomogeneous index of refraction subset of a larger domain Ω . and D, Ω are C^1 domains. Second, we assume $a_{ij}(x) = a_{ij}^0(x) \chi_{\Omega \setminus D} + \widetilde{a}_{ij}(x) \chi_D$, where $\widetilde{a}_{ij}(x)$ is regarded as a perturbation in the unknown obstacle D and $\widetilde{a}_{ij}(x)$ satisfies $\widetilde{\lambda} |\xi|^2 \leq \sum_{i,j} \widetilde{a}_{ij}(x) \xi_i \xi_j \leq \widetilde{\Lambda} |\xi|^2$. Moreover, we need to assume that there exists a universal constant $0 < \widehat{\lambda} \leq \widehat{\Lambda}$ such that $\forall \xi \in \mathbb{R}^3$, we have $\widehat{\lambda} |\xi|^2 \leq \sum (\widetilde{a}_{ij}(x) \chi_D - a_{ij}^0(x)) \xi_i \xi_j \leq \widehat{\Lambda} |\xi|^2$, which mean the perturbed term $\widetilde{A}(x)$ is “greater” than the unperturbed term A^0 inside the unknown obstacle D . Denote $A(x) = (a_{ij}(x))$, $A^0(x) = (a_{ij}^0(x))$ and let $k > 0$ and consider the steady state anisotropic acoustic wave equation in with

Dirichlet boundary condition

$$\begin{cases} \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (1.1)$$

In the unperturbed case, we have

$$\begin{cases} \nabla \cdot (A^0(x)\nabla u_0) + k^2 u_0 = 0 & \text{in } \Omega \\ u_0 = f & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

In this paper, we assume that k^2 is not a Dirichlet eigenvalue of the operator $-\nabla \cdot (A\nabla \bullet)$ and $-\nabla \cdot (A^0\nabla \bullet)$ in Ω . It is known that for any $f \in H^{1/2}(\partial\Omega)$, there exists a unique solution u to (1.1). We define the Dirichlet-to-Neumann map in the anisotropic case, say $\Lambda_D : H^{1/2}(\partial\Omega) \rightarrow H^{-1/2}(\partial\Omega)$ as the following.

Definition 1.1. $\Lambda_D f := A\nabla u \cdot \nu = \sum_{i,j=1}^3 a_{ij} \partial_j u \cdot \nu_i$ and $\Lambda_\emptyset f := A^0\nabla u_0 \cdot \nu = \sum_{i,j=1}^3 a_{ij} \partial_j u_0 \cdot \nu_i$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is an outer normal on $\partial\Omega$.

Inverse problem: Identify the location and the convex hull of D from the DN-map Λ_D . The domain D can also be treated as an inclusion embedded in Ω . The aim of this work is to give a reconstruction algorithm for this problem. Note that the information on the medium parameter $(\widetilde{a_{ij}}(x))$ inside D is not known a priori.

The main tool in our reconstruction method is the oscillating-decaying solutions for the second order anisotropic elliptic differential equations. We use the results coming from the paper [2] to construct the oscillating-decaying solution. In section 2, we will construct the oscillating-decaying solutions for anisotropic elliptic equations, note that even if $k = 0$, which means the equation is $\nabla \cdot (A(x)\nabla u) = 0$, we do not have any CGO-type solutions. Roughly speaking, given a hyperplane, an oscillating-decaying solution is oscillating very rapidly along this plane and decaying exponentially in the direction transversely to the same plane. They are also CGO-solutions but with the imaginary part of the phase function non-negative. Note that the domain of the oscillating-decaying solutions is not over the whole Ω , so we need to extend such solutions to the whole domain. Fortunately, the Runge approximation property provides us a good approach to extend this special solution in section 3.

In Ikehata's work, the CGO-solutions are used to define the indicator function (see [10] for the definition). In order to use the oscillating-decaying solutions to the inverse problem of identifying an inclusion, we have to modify the definition of the indicator function using the Runge approximation property. It was first recognized by Lax [1] that the Runge approximation property is a consequence of the weak unique continuation property. In our case, it is clear that the anisotropic elliptic equation has the weak unique continuation property if the leading part is Lipschitz continuous.

2 Construction of oscillating-decaying solutions

In this section, we follow the paper [2] to construct the oscillating-decaying solution in the anisotropic elliptic equations. In our case, since we only consider

a scalar elliptic equation, it's construction is simpler than the construction in [2]. Consider the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Note that the oscillating-decaying solutions of

$$\begin{cases} \nabla \cdot (A(x)\nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

will have the same representation as the equation (2.1), that is, the lower order term $k^2 u$ will not affect the form of the oscillating-decaying solutions, we will see the detail in the following constructions. Now, we assume that the domain Ω is an open, bounded smooth domain in \mathbb{R}^3 and the coefficients $A(x) = (a_{ij}(x))$ satisfying $\sum_{i,j=1}^3 a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$, $\forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ and λ is a universal constant.

Assume that

$$A(x) = (a_{ij}(x)) \in B^\infty(\mathbb{R}^3) = \{f \in C^\infty(\mathbb{R}^3) : \partial^\alpha f \in L^\infty(\mathbb{R}^3), \forall \alpha \in \mathbb{Z}_+^3\}$$

is the anisotropic coefficients satisfying $a_{ij}(x) = a_{ji}(x) \forall i, j$ and there exists a $\lambda > 0$ such that $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \forall x \in \mathbb{R}^3$ (uniform ellipticity). It is clear that $A(x)$ is Lipschitz continuous if each $a_{ij}(x) \in B^\infty(\mathbb{R}^3)$, it has weak continuation property.

We give several notations as follows. Assume that $\Omega \subset \mathbb{R}^3$ is an open set with smooth boundary and $\omega \in S^2$ is given. Let $\eta \in S^2$ and $\zeta \in S^2$ be chosen so that $\{\eta, \zeta, \omega\}$ forms an orthonormal system of \mathbb{R}^3 . We then denote $x' = (x \cdot \eta, x \cdot \zeta)$. Let $t \in \mathbb{R}$, $\Omega_t(\omega) = \Omega \cap \{x \cdot \omega > t\}$ and $\Sigma_t(\omega) = \Omega \cap \{x \cdot \omega = t\}$ be a non-empty open set. We consider a scalar function $u_{\chi_t, t, b, N, \omega}(x, \tau) := u(x, \tau) \in C^\infty(\overline{\Omega_t(\omega)} \setminus \partial\Sigma_t(\omega)) \cap C^0(\overline{\Omega_t(\omega)})$ with $\tau \gg 1$ satisfying:

$$\begin{cases} L_A u = \nabla \cdot (A(x)\nabla u) + k^2 u = 0 & \text{in } \Omega_t(\omega) \\ u = e^{i\tau x \cdot \xi} \{\chi_t(x') Q_t(x') b + \beta_{\chi_t, t, b, N, \omega}\} & \text{on } \Sigma_t(\omega), \end{cases} \quad (2.2)$$

where $\xi \in S^2$ lying in the span of η and ζ is chosen and fixed, $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ with $\text{supp}(\chi_t) \subset \Sigma_t(\omega)$, $Q_t(x')$ is a nonzero smooth function and $0 \neq b \in \mathbb{C}$. Moreover, $\beta_{\chi_t, b, t, N, \omega}(x', \tau)$ is a smooth function supported in $\text{supp}(\chi_t)$ satisfying:

$$\|\beta_{\chi_t, b, t, N, \omega}(\cdot, \tau)\|_{L^2(\mathbb{R}^2)} \leq c\tau^{-1}$$

for some constant $c > 0$. From now on, we use c to denote a general positive constant whose value may vary from line to line. As in the paper [2], $u_{\chi_t, b, t, N, \omega}$ can be written as

$$u_{\chi_t, b, t, N, \omega} = w_{\chi_t, b, t, N, \omega} + r_{\chi_t, b, t, N, \omega}$$

with

$$w_{\chi_t, b, t, N, \omega} = \chi_t(x') Q_t e^{i\tau x \cdot \xi} e^{-\tau(x \cdot \omega - t) A_t(x')} b + \gamma_{\chi_t, b, t, N, \omega}(x, \tau) \quad (2.3)$$

and $r_{\chi_t b, t, N, \omega}$ satisfying

$$\|r_{\chi_t b, t, N, \omega}\|_{H^1(\Omega_t(\omega))} \leq c\tau^{-N-1/2}, \quad (2.4)$$

where $A_t(\cdot) \in B^\infty(\mathbb{R}^2)$ is a complex function with its real part $\operatorname{Re} A_t(x') > 0$, and $\gamma_{\chi_t b, t, N, \omega}$ is a smooth function supported in $\operatorname{supp}(\chi_t)$ satisfying

$$\|\partial_x^\alpha \gamma_{\chi_t b, t, N, \omega}\|_{L^2(\Omega_s(\omega))} \leq c\tau^{|\alpha|-3/2} e^{-\tau(s-t)a} \quad (2.5)$$

for $|\alpha| \leq 1$ and $s \geq t$, where $a > 0$ is some constant depending on $A_t(x')$.

Without loss of generality, we consider the special case where $t = 0$, $\omega = e_3 = (0, 0, 1)$ and choose $\eta = (1, 0, 0)$, $\zeta = (0, 1, 0)$. The general case can be obtained from this special case by change of coordinates. Define $L = L_A$ and $\widetilde{M} = e^{-i\tau x' \cdot \xi'} L(e^{i\tau x' \cdot \xi'} \cdot)$, where $x' = (x_1, x_2)$ and $\xi' = (\xi_1, \xi_2)$ with $|\xi'| = 1$, then \widetilde{M} is a differential operator. To be precise, by using $a_{jl} = a_{lj}$, we calculate \widetilde{M} to be given by

$$\begin{aligned} \widetilde{M} &= -\tau^2 \sum_{jl} a_{jl} \xi_j \xi_l + 2\tau \sum_{jl} a_{jl} (i\xi_l) \partial_j + \sum_{jl} a_{jl} \partial_j \partial_l \\ &\quad + \sum_{jl} (\partial_j a_{jl}) (i\tau \xi_l) + \sum_{jl} (\partial_l a_{jl}) \partial_l + k^2 \\ &= -\tau^2 \sum_{jl} a_{jl} \xi_j \xi_l + 2\tau \sum_l a_{3l} (i\xi_l) \partial_3 + a_{33} \partial_3 \partial_3 \\ &\quad + 2\tau \sum_{j \neq 3, l} a_{jl} (i\xi_l) \partial_j + \sum_{(j, l) \setminus \{3, 3\}} a_{jl} \partial_j \partial_l \\ &\quad + \sum_{jl} (\partial_j a_{jl}) (i\tau \xi_l) + \sum_{jl} (\partial_l a_{jl}) \partial_l + k^2 \end{aligned}$$

with $\xi_3 = 0$. Now, we want to solve

$$\widetilde{M}v = 0,$$

which is equivalent to $Mv = 0$, where $M = a_{33}^{-1} \widetilde{M}$. Now, we use the same idea in [2], define $\langle e, f \rangle = \sum_{ij} a_{ij} e_i f_j$, where $e = (e_1, e_2, e_3)$, $f = (f_1, f_2, f_3)$ and denote $\langle e, f \rangle_0 = \langle e, f \rangle|_{x_3=0}$. Let P be a differential operator, and we define the order of P , denoted by $\operatorname{ord}(P)$, in the following sense:

$$\|P(e^{-\tau x_3 A(x')} \varphi(x'))\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{\operatorname{ord}(P)-1/2},$$

where $\mathbb{R}_+^3 = \{x_3 > 0\}$, $A(x')$ is a smooth complex function with its real part greater than 0 and $\varphi(x') \in C_0^\infty(\mathbb{R}^2)$. In this sense, similar to [2], we can see that τ , ∂_3 are of order 1, ∂_1, ∂_2 are of order 0 and x_3 is of order -1.

Now according to this order, the principal part M_2 (order 2) of M is:

$$M_2 = -\{D_3^2 + 2\tau \langle e_3, e_3 \rangle_0^{-1} \langle e_3, \rho \rangle_0 D_3 + \tau^2 \langle e_3, e_3 \rangle_0^{-1} \langle \rho, \rho \rangle_0\}$$

with $D_3 = -i\partial_3$ and $\rho = (\xi_1, \xi_2, 0)$. Note that the principal part M_2 does not involve the lower order term k^2 . Note that M_2 is obtained by the Taylor's expansion of M at $x_3 = 0$, that is,

$$\begin{aligned} M(x', x_3) &= M(x', 0) + x_3 \partial_3 M(x', 0) + \cdots + \frac{x_3^{N-1}}{(N-1)!} \partial_3^{N-1} M(x', 0) + R \\ &= M_2 + M_1 + \cdots + M_{-N+1} + R, \end{aligned}$$

where $\text{ord}(M_j) = j$ and $\text{ord}(R) = -N$. To solve $Mv = 0$ is equivalent to solve

$$M_2 v = -(M_1 + \cdots + M_{-N+1} + R)v := f. \quad (2.6)$$

If we set $w_1 = v$ and $w_2 = -\tau^{-1} \langle e_3, e_3 \rangle_0 D_3 v - \langle e_3, \rho \rangle_0 v$, then we can compute

$$D_3 w_1 = -\tau \langle e_3, e_3 \rangle_0^{-1} \langle e_3, \rho \rangle_0 w_1 - \tau \langle e_3, e_3 \rangle_0^{-1} w_2 \quad (2.7)$$

and

$$\begin{aligned} D_3 w_2 &= -\tau \{ \langle \rho, e_3 \rangle_0^2 \langle e_3, e_3 \rangle_0^{-1} - \langle \rho, \rho \rangle_0 \} w_1 - \tau \langle \rho, e_3 \rangle_0 \langle e_3, e_3 \rangle_0^{-1} w_2 \\ &\quad + \tau^{-1} \langle e_3, e_3 \rangle_0 f. \end{aligned} \quad (2.8)$$

For detail calculations, we refer readers to see [2]. If we set $W = [w_1, w_2]^T$ and use (2.7) and (2.8), we have

$$D_3 W = \tau K W + \begin{bmatrix} 0 \\ \tau^{-1} \langle e_3, e_3 \rangle_0 f \end{bmatrix},$$

where

$$K = \begin{bmatrix} \langle e_3, e_3 \rangle_0^{-1} \langle e_3, \rho \rangle_0 & \langle e_3, e_3 \rangle_0^{-1} \\ \langle \rho, e_3 \rangle_0^2 \langle e_3, e_3 \rangle_0^{-1} - \langle \rho, \rho \rangle_0 & \langle \rho, e_3 \rangle_0 \langle e_3, e_3 \rangle_0^{-1} \end{bmatrix}. \quad (2.9)$$

By (2.6), we can express (2.9) as

$$D_3 W = (\tau K + K_0 + \cdots + K_{-N} + S)W, \quad (2.10)$$

where $\text{ord}(K_j) = j$ and $\text{ord}(S) = -N - 1$ and all the differential operators K_j involves only x' derivatives. Moreover, K is a matrix function independent of x_3 and its eigenvalues are determined from

$$\det(\lambda I - K) = 0,$$

which is equivalent to

$$\lambda^2 - 2 \langle e_3, e_3 \rangle_0^{-1} \langle e_3, \rho \rangle_0 \lambda + \langle \rho, \rho \rangle_0 = 0. \quad (2.11)$$

By using the uniform elliptic assumption on (a_{ij}) that (2.11) has roots λ^\pm with $\text{Im} \lambda^\pm > 0$. Similar to [2], we can set $\tilde{Q} = [q^+, q^-]$ be a nonsingular matrix with linearly independent vectors q^\pm such that

$$\tilde{K} = \tilde{Q}^{-1} K \tilde{Q} = \begin{bmatrix} \lambda^+ & 0 \\ 0 & \lambda^- \end{bmatrix},$$

where $\lambda^\pm \in \mathbb{C}_\pm := \{\pm \text{Im} \lambda > 0\}$, respectively. Moreover, we choose

$$\tilde{Q} = \begin{bmatrix} q & \bar{q} \\ q' & \bar{q}' \end{bmatrix}, \quad (2.12)$$

where

$$\begin{bmatrix} q \\ q' \end{bmatrix} = [q^+] \text{ and } \begin{bmatrix} \bar{q} \\ \bar{q}' \end{bmatrix} = [q^-]$$

By virtue of the matrix \tilde{Q} in (2.12), we have $\lambda^- = \overline{\lambda^+}$, and \tilde{Q} is nonsingular. If we set $\widehat{Q} = \tilde{Q}^{-1}W$, we get from (2.10) that

$$D_3\widehat{W} = (\tau\tilde{K} + \widehat{K}_0 + \cdots + \widehat{K}_{-N} + S)\widehat{W}, \quad (2.13)$$

where $\text{ord}(\widehat{K}_j) = j$ and $\text{ord}(\widehat{S}) = -N - 1$. Similar as before, we know that \widehat{K}_j contains only x' derivatives since the original K_j involves only x' derivatives. In addition, \widehat{K}_0 can be divided into terms involving τx_3 and terms formed by the differential operator in $\partial_{x'}$ with coefficients independent of x_3 . Likewise, \widehat{K}_j can be grouped into terms containing $\tau x_3^{-j+1}, \tau^{-1}x_3^{j-1}, x_3^{-j}$, respectively, where $-N \leq j \leq -1$.

From now on, we have decoupled K by choosing a suitable matrix function \tilde{Q} , next we want to decouple $\widehat{K}_0, \dots, \widehat{K}_{-N}$. First, we show how to decouple \widehat{K}_0 . Let $\widehat{W} = (1 + x_3 A^{(0)} + \tau^{-1} B^{(0)})\widetilde{W}^{(0)}$ with $A^{(0)}, B^{(0)}$ being differential operators in $\partial_{x'}$ with coefficients independent of x_3 , then we have

$$\begin{aligned} D_3\widehat{W}^{(0)} &= \{\tau\tilde{K} + (\widehat{K}_0 - \tau x_3 A^{(0)}\tilde{K} + \tau x_3 \tilde{K} A^{(0)} - B^{(0)}\tilde{K} + \tilde{K} B^{(0)} + iA^{(0)}) \\ &\quad + \widehat{K}'_{-1} + \cdots\}\widehat{W}^{(0)}, \end{aligned}$$

where $\text{ord}(\widehat{K}'_{-1}) = -1$ and the remainder contains terms of order at most -2. Let $\tilde{K}_0 := \widehat{K}_0 - \tau x_3 A^{(0)}\tilde{K} + \tau x_3 \tilde{K} A^{(0)} - B^{(0)}\tilde{K} + \tilde{K} B^{(0)} + iA^{(0)}$, we analyze \tilde{K}_0 more carefully. Set $\tilde{K}_0 = \tau x_3 \tilde{K}_{0,1} + \tilde{K}_{0,2}$ and express $\tilde{K}_{0,1}, \tilde{K}_{0,2}, A^{(0)}$ and $B^{(0)}$ in block forms, that is,

$$\begin{aligned} \tilde{K}_{0,l} &= \begin{bmatrix} \widehat{K}_{0,l}(1,1) & \widehat{K}_{0,l}(1,2) \\ \widehat{K}_{0,l}(2,1) & \widehat{K}_{0,l}(2,2) \end{bmatrix}, \quad l = 1, 2, \\ A^{(0)} &= \begin{bmatrix} A^{(0)}(1,1) & A^{(0)}(1,2) \\ A^{(0)}(2,1) & A^{(0)}(2,2) \end{bmatrix} \quad \text{and} \quad B^{(0)} = \begin{bmatrix} B^{(0)}(1,1) & B^{(0)}(1,2) \\ B^{(0)}(2,1) & B^{(0)}(2,2) \end{bmatrix}. \end{aligned}$$

Then the off-diagonal blocks of \tilde{K}_0 are given by:

$$\begin{aligned} \tilde{K}_0(1,2) &= \tau x_3 \{\widehat{K}_{0,1}(1,2) - A^{(0)}(1,2)\lambda^- + \lambda^+ A^{(0)}(1,2)\} \\ &\quad + \{\widehat{K}_{0,2}(1,2) + iA^{(0)}(1,2) - B^{(0)}(1,2)\lambda^- + \lambda^+ B^{(0)}(1,2)\}, \end{aligned}$$

$$\begin{aligned} \tilde{K}_0(2,1) &= \tau x_3 \{\widehat{K}_{0,1}(2,1) - A^{(0)}(2,1)\lambda^- + \lambda^+ A^{(0)}(2,1)\} \\ &\quad + \{\widehat{K}_{0,2}(2,1) + iA^{(0)}(2,1) - B^{(0)}(2,1)\lambda^- + \lambda^+ B^{(0)}(2,1)\}. \end{aligned}$$

Since $\lambda^\pm \in \mathbb{C}_\pm$, we can find suitable $A^{(0)}(1,2)$ and $A^{(0)}(2,1)$ such that

$$\begin{cases} \widehat{K}_{0,1}(1,2) - A^{(0)}(1,2)\lambda^- + \lambda^+ A^{(0)}(1,2) = 0 \\ \widehat{K}_{0,1}(2,1) - A^{(0)}(2,1)\lambda^- + \lambda^+ A^{(0)}(2,1) = 0 \end{cases}$$

(see similar arguments in [13]). Similarly, we can use the same method to find $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$ so that

$$\begin{cases} \widehat{K}_{0,2}(1,2) + iA^{(0)}(1,2) - B^{(0)}(1,2)\lambda^- + \lambda^+ B^{(0)}(1,2) = 0, \\ \widehat{K}_{0,2}(2,1) + iA^{(0)}(2,1) - B^{(0)}(2,1)\lambda^- + \lambda^+ B^{(0)}(2,1) = 0. \end{cases} \quad (2.14)$$

Since $\widehat{K}_{0,2}(1,2)$ and $\widehat{K}_{0,2}(2,1)$ are differential operators in $\partial_{x'}$ with coefficients independent of x_3 , we will look for $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$ as the same type of differential operators. By (2.14) and using $\lambda^\pm \in \mathbb{C}_\pm$, we can solve for $B^{(0)}(1,2)$ and $B^{(0)}(2,1)$. To find $A^{(0)}$ and $B^{(0)}$, we simply set diagonal blocks of them are zero, i.e.,

$$A^{(0)} = \begin{bmatrix} 0 & A^{(0)}(1,2) \\ A^{(0)}(2,1) & 0 \end{bmatrix} \text{ and } B^{(0)} = \begin{bmatrix} 0 & B^{(0)}(1,2) \\ B^{(0)}(2,1) & 0 \end{bmatrix}.$$

With these matrices $A^{(0)}$ and $B^{(0)}$, we can see that

$$D_3 \widetilde{W}^{(0)} = \{\tau \widetilde{K} + \widetilde{K}_0 + \widetilde{K}'_{-1} + \dots\} \widetilde{W}^{(0)} \quad (2.15)$$

where

$$\widetilde{K}_0 = \begin{bmatrix} \widetilde{K}_0(1,1) & 0 \\ 0 & \widetilde{K}_0(2,2) \end{bmatrix}.$$

Moreover, we want to decouple \widehat{K}'_{-1} and \widehat{K}'_{-1} can be written as $\widehat{K}'_{-1} = \tau x_3^2 \widehat{K}'_{-1,1} + x_3 \widehat{K}'_{-1,2} + \tau^{-1} \widehat{K}'_{-1,3}$. We can see that $\widehat{K}'_{-1,1}$, $\widehat{K}'_{-1,2}$ and $\widehat{K}'_{-1,3}$ are differential operators in $\partial_{x'}$ of order zero, one and two with coefficients independent of x_3 , respectively. Similarly, we can set $\widetilde{W}^{(0)} = (I + x_3^2 A^{(1)} + \tau^{-1} x_3 B^{(1)} + \tau^{-2} C^{(1)}) \widetilde{W}^{(1)}$, where $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ are differential operators in $\partial_{x'}$. Now plugging $\widetilde{W}^{(0)}$ of above form into (2.15), we have

$$\begin{aligned} D_3 \widetilde{W}^{(1)} &= \{\tau \widetilde{K} + \widetilde{K}_0 + \tau x_3^2 (\widehat{K}'_{-1,1} - A^{(1)} \widetilde{K} + \widetilde{K} A^{(1)}) + x_3 (\widehat{K}'_{-1,2} - B^{(1)} \widetilde{K} \\ &\quad + \widetilde{K} B^{(1)} + 2A^{(1)}) + \tau^{-1} (\widehat{K}'_{-1,3} - C^{(1)} \widetilde{K} + \widetilde{K} C^{(1)} + iB^{(1)}) \\ &\quad + \dots\} \widetilde{W}^{(1)} \end{aligned} \quad (2.16)$$

where the remainder consists of terms with order at most -2. Then we use the same argument, we can find suitable $A^{(1)}$, $B^{(1)}$ and $C^{(1)}$ such that the off-diagonal blocks of the order -1 term on the right hand side of (2.16) are zero. Therefore, we obtain

$$D_3 \widetilde{W}^{(1)} = \{\tau \widetilde{K} + \widetilde{K}_0 + \widetilde{K}'_{-1} + \dots\} \widetilde{W}^{(1)}$$

with

$$\widetilde{K}'_{-1} = \begin{bmatrix} \widetilde{K}'_{-1}(1,1) & 0 \\ 0 & \widetilde{K}'_{-1}(2,2) \end{bmatrix}.$$

Recursively, by defining

$$\begin{aligned} \widehat{W} &= (I + x_3 A^{(0)} + \tau^{-1} B^{(0)}) (I + x_3^2 A^{(1)} + \tau^{-1} x_3 B^{(1)} + \tau^{-2} C^{(1)}) \dots \\ &\quad (I + x_3^{N+1} A^{(N)} + \tau^{-1} x_3^N B^{(N)} + \tau^{-2} x_3^{N-1} C^{(N)}) \widetilde{W}^{(N)} \end{aligned}$$

with suitable $A^{(j)}$, $B^{(j)}$ and $C^{(j)}$ for $0 \leq j \leq N$ ($C^{(0)} = 0$), we can transform the equation (2.13) into

$$D_3 \widetilde{W}^{(N)} = \{\tau \widetilde{K} + \widetilde{K}_0 + \dots + \widetilde{K}'_{-N} + \widetilde{S}\} \widetilde{W}^{(N)}, \quad (2.17)$$

where \widetilde{K}'_{-j} for all $0 \leq j \leq N$ are decoupled and $\text{ord}(\widetilde{S}) = -N - 1$. Note that all diagonal blocks of $A^{(j)}$ and $B^{(j)}$ are zero.

Now in view of (2.17), we consider the equation

$$D_3 \hat{v}^{(N)} = \{\tau \lambda^+ + \tilde{K}_0(1, 1) + \cdots + \tilde{K}_{-N}(1, 1)\} \hat{v}^{(N)},$$

with an approximated solution of the form

$$\hat{v}^{(N)} = \sum_{j=0}^{N+1} \hat{v}_{-j}^{(N)},$$

where $\hat{v}_{-j}^{(N)}$ for $0 \leq j \leq N$ satisfy

$$\begin{cases} D_3 \hat{v}_0^{(N)} = \tau \lambda^+ \hat{v}_0^{(N)}, & \hat{v}_0^{(N)}|_{x_3=0} = \chi_t(x')b \\ D_3 \hat{v}_{-1}^{(N)} = \tau \lambda^+ \hat{v}_{-1}^{(N)} + \tilde{K}_0(1, 1) \hat{v}_0^{(N)}, & \hat{v}_{-1}^{(N)}|_{x_3=0} = 0 \\ \vdots & \vdots \\ D_3 \hat{v}_{-N-1}^{(N)} = \tau \lambda^+ \hat{v}_{-N-1}^{(N)} + \sum_{j=0}^N \tilde{K}_{-j}(1, 1) \hat{v}_{-j}^{(N)}, & \hat{v}_{-N-1}^{(N)}|_{x_3=0} = 0, \end{cases}$$

where $\chi_t(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}$. It is easy to solve $\hat{v}_0^{(N)} = \exp(i\tau x_3 \lambda^+) \chi_t(x')b$ and $\hat{v}_{-1}^{(N)} = \exp(i\tau x_3 \lambda^+) \int_0^{x_3} \exp(-i\tau s \lambda^+) \tilde{K}_0(1, 1) \hat{v}_0^{(N)} ds$. Moreover, we can use the $\text{ord}(x_3) = -1$ and $\text{ord}(\partial_j) = 0$ with $j = 1, 2$ to derive that

$$\|x_3^\beta \partial_{x'}^\alpha \hat{v}_0^{(N)}\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta-1/2}$$

for $\beta \in \mathbb{Z}_+$ and multi-index α . Similarly, we can compute

$$\|\hat{v}_{-1}^{(N)}\|_{L^2(\mathbb{R}_+^3)}^2 \leq c\tau^{-3}. \quad (2.18)$$

For the derivation of (2.18), it can be found in [2]. Moreover, by similar computations we can show that

$$\|x_3^\beta \partial_{x'}^\alpha (\hat{v}_{-1}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta-3/2}$$

and for $\hat{v}_{-j}^{(N)}$, $j = 2, \dots, N+1$, we have

$$\|x_3^\beta \partial_{x'}^\alpha (\hat{v}_{-j}^{(N)})\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-\beta-j-1/2}$$

for $2 \leq j \leq N+1$.

Thus, if we set $V^{(N)} = \begin{bmatrix} \hat{v}^{(N)} \\ 0 \end{bmatrix}$, then we have

$$\begin{cases} D_3 V^{(N)} - \{\tau \tilde{K} + \tilde{K}_0 + \cdots + \tilde{K}_{-N}\} V^{(N)} = \tilde{R}, \\ V^{(N)}|_{x_3=0} = \begin{bmatrix} \chi_t(x')b \\ 0 \end{bmatrix}, \end{cases}$$

where

$$\|\tilde{R}\|_{L^2(\mathbb{R}_+^3)} \leq c\tau^{-N-3/2}.$$

Define v to be the function of the first component of $\tilde{Q}(I + x_3 A^{(0)} + \tau^{-1} B^{(0)})(I + x_3^2 A^{(1)} + \tau^{-1} x_3 B^{(1)} + \tau^{-2} C^{(1)}) \cdots (I + x_3^{N+1} A^{(N)} + \tau^{-1} x_3^N B^{(N)} + \tau^{-2} x_3^{N-1} C^{(N)}) V^{(N)}$ and set $w = \exp(i\tau x' \cdot \xi') \tilde{v}$, we have

$$\begin{aligned} w &= q \exp(i\tau x' \cdot \xi') \exp(i\tau x_3 \lambda^+(x')) \chi_t(x')b + \exp(i\tau x' \cdot \xi') \tilde{\gamma}(x, \tau) \\ &\quad q \exp(i\tau x' \cdot \xi') \exp(-i\tau x_3 (-i\lambda^+(x'))) \chi_t(x')b + \gamma(x, \tau) \end{aligned}$$

and

$$w|_{x_3=0} = \exp(i\tau x' \cdot \xi') \{ \chi_t(x') q b + \beta_0(x', \tau) \},$$

where γ satisfies the estimate (2.5) on $\Omega_s := \{x_3 > s\} \cap \Omega$ for $s \geq 0$ and $\beta_0(x', \tau) = \tilde{\gamma}(x', 0, \tau)$ is supported in $\text{supp}(\chi_t)$ with $\|\beta_0(\cdot, \tau)\|_{L^\infty} \leq c\tau^{-1}$. Also, we have

$$\|M\tilde{v}\|_{L^2(\Omega_0)} \leq c\tau^{-N-1/2}.$$

Let $u = w + r = e^{ix' \cdot \xi'} \tilde{v} + r$ and r be the solution to the boundary value problem

$$\begin{cases} Lr = -e^{ix' \cdot \xi'} \widetilde{M\tilde{v}} & \text{in } \Omega_0, \\ r = 0 & \text{on } \partial\Omega_0. \end{cases} \quad (2.19)$$

The existence of r solving (2.19) is by using the Lax-Milgram theorem and we have the following estimate

$$\|r\|_{H^1(\Omega_0)} \leq c\tau^{-N-1/2},$$

which is the estimate (2.4) on Ω_0 . We complete the construction of the oscillating-decaying solutions for the case $t = 0$ and $\omega = (0, 0, 1)$ in the anisotropic elliptic equations case. The oscillating-decaying solution in the general case can be obtained by using change of coordinates.

3 Tools and estimates

In this section, we introduce the Runge approximation property and a very useful elliptic estimate: Meyers L^p -estimates.

3.1 Runge approximation property

Definition 3.1. [1] Let L be a second order elliptic operator, solutions of an equation $Lu = 0$ are said to have the Runge approximation property if, whenever K and Ω are two simply connected domains with $K \subset \Omega$, any solution in K can be approximated uniformly in compact subsets of K by a sequence of solutions which can be extended as solution to Ω .

There are many applications for Runge approximation property in inverse problems. Similar results for some elliptic operators can be found in [1], [14]. The following theorem is a classical result for Runge approximation property for a second order elliptic equation.

Theorem 3.2. (*Runge approximation property*) Let $L_0 \cdot = \nabla(A^0(x)\nabla \cdot) + k^2 \cdot$ be a second order elliptic differential operator with $A^0(x)$ to be Lipschitz. Assume that k^2 is not a Dirichlet eigenvalue of $-\nabla(A^0(x)\nabla \cdot)$. Let O and Ω be two open bounded domains with smooth boundary in \mathbb{R}^3 such that O is convex and $\bar{O} \subset \Omega$.

Let $u_0 \in H^1(O)$ satisfy

$$L_0 u_0 = 0 \text{ in } O.$$

Then for any compact subset $K \subset O$ and any $\epsilon > 0$, there exists $U \in H^1(\Omega)$ satisfying

$$L_0 U = 0 \text{ in } \Omega,$$

such that

$$\|u_0 - U\|_{H^1(K)} \leq \epsilon.$$

Note that we have assumed that $A^0 \in B^\infty(\mathbb{R}^3)$, it is easy to see $A^0(x)$ is a Lipschitz continuous function, it possesses the weak continuation property. The proof can be found in [1] and [2], we omit details here.

3.2 Elliptic estimates and some identities

We need some estimates for solutions to some Dirichlet problems which will be used in next section. Recall that, for $f \in H^{1/2}(\partial\Omega)$, let u and u_0 be solutions to the Dirichlet problems (1.1) and (1.2), respectively. Note that $a_{ij}(x) = a_{ij}^0(x)\chi_{\Omega \setminus D} + \widetilde{a_{ij}}(x)\chi_D$ and we set $w = u - u_0$, then w satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x)\nabla w) + k^2 w = -\nabla \cdot ((\widetilde{A}\chi_D - A^0\chi_D)\nabla u_0) & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \quad (3.1)$$

where $A(x) = (a_{ij}(x))$, $A^0(x) = (a_{ij}^0(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. Then we have some estimates for w .

Lemma 3.3. *There exists a positive constant C independent of w such that we have*

$$\|w\|_{L^2(\Omega)} \leq C\|\nabla w\|_{L^p(\Omega)}$$

for $\frac{6}{5} \leq p \leq 2$ if $n = 3$.

The proof follow from [6] by Freidrichs inequality, see [4] p.258 and use a standard elliptic regularity.

Lemma 3.4. *There exists $\epsilon \in (0, 1)$, depending only on Ω , $A^0(x) = (a_{ij}^0(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$ such that*

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2 - \epsilon, \frac{6}{5}\} < p \leq 2$ if $n = 3$.

Proof. The proof is also followed from [6]. Set $f := -(\widetilde{A}\chi_D - A^0\chi_D)\nabla u_0$, $h := 0$. Let w_0 be a solution of

$$\begin{cases} \nabla \cdot (A(x)\nabla w_0) + k^2 w_0 = \nabla \cdot f & \text{in } \Omega, \\ w_0 = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

The following L^p -estimate of w_0 , followed from [5], then we can get

$$\|\nabla w_0\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \quad (3.3)$$

for $p \in (\max\{2 - \epsilon, \frac{6}{5}\}, 2]$, where $\epsilon \in (0, 1)$ depends on Ω , $A^0(x) = (a_{ij}^0(x))$ and $\widetilde{A}(x) = (\widetilde{a_{ij}}(x))$. We set $W := w - w_0$, then since $w = w_0 + W$, we have

$$\|\nabla w\|_{L^p(\Omega)} \leq C(\|\nabla w_0\|_{L^p(\Omega)} + \|\nabla W\|_{L^p(\Omega)}). \quad (3.4)$$

Moreover, W satisfies

$$\begin{cases} \nabla \cdot (A(x)\nabla W) + k^2 W = 0 & \text{in } \Omega, \\ W = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.5)$$

By the standard elliptic regularity, we have

$$\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}.$$

Thus, we get for $p \leq 2$,

$$\|\nabla W\|_{L^p(\Omega)} \leq C\|\nabla W\|_{L^2(\Omega)} \leq C\|W\|_{H^1(\Omega)} \leq C\|w_0\|_{L^2(\Omega)}. \quad (3.6)$$

By Sobolev embedding theorem, we get

$$\|w_0\|_{L^2(\Omega)} \leq C\|w_0\|_{W^{1,p}(\Omega)} \quad (3.7)$$

for $p \geq \frac{6}{5}$ if $n = 3$. Use Poincaré's inequality in L^p spaces ($w_0|_{\partial\Omega} = 0$), we have

$$\|w_0\|_{L^2(\Omega)} \leq C\|\nabla w_0\|_{L^p(\Omega)} \quad (3.8)$$

for $p \geq \frac{6}{5}$ if $n = 3$. Combining (3.3) with (3.4), (3.6) and (3.8), we can obtain

$$\|\nabla w\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$$

for $\max\{2 - \epsilon, \frac{6}{5}\} < p \leq 2$ if $n = 3$. \square

Recall the Dirichlet-to-Neumann map which we have defined in the section 1: $\Lambda_D f := A\nabla u \cdot \nu$ and $\Lambda_\emptyset f := A^0\nabla u_0 \cdot \nu$, where $\nu = (\nu_1, \nu_2, \nu_3)$ is an outer normal on $\partial\Omega$. We next prove some useful identities.

Lemma 3.5. $\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f \bar{f} d\sigma = \operatorname{Re} \int_D (\tilde{A} - A^0) \nabla u_0 \cdot \overline{\nabla u} dx.$

Proof. It is clear that

$$\begin{aligned} \int_{\partial\Omega} A\nabla u \cdot \nu \bar{\varphi} d\sigma &= \int_{\Omega} \nabla \cdot (A\nabla u \bar{\varphi}) dx \\ &= \int_{\Omega} \nabla \cdot (A\nabla u) \bar{\varphi} + A\nabla u \cdot \overline{\nabla \varphi} dx \\ &= -k^2 \int_{\Omega} u \bar{\varphi} dx + \int_{\Omega} A\nabla u \cdot \overline{\nabla \varphi} dx \end{aligned}$$

$\forall \varphi \in H^1(\Omega)$. Since $u = u_0 = f$ on $\partial\Omega$, the left hand side of the identity has the same value whether we take $\varphi = u$ or $\varphi = u_0$, and it is equal to $\int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma$.

$$\begin{aligned} \int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma &= -k^2 \int_{\Omega} u \overline{u_0} dx + \int_{\Omega} A\nabla u \cdot \overline{\nabla u_0} dx \\ &= -k^2 \int_{\Omega} |u|^2 dx + \int_{\Omega} A\nabla u \cdot \overline{\nabla u} dx. \end{aligned}$$

The right hand side of the identity above is real. Hence, by taking the real part, we have

$$\int_{\partial\Omega} \Lambda_D f \bar{f} d\sigma = -k^2 \operatorname{Re} \int_{\Omega} u \overline{u_0} dx + \operatorname{Re} \int_{\Omega} A \nabla u \cdot \overline{\nabla u_0} dx$$

and

$$\int_{\partial\Omega} \Lambda_{\emptyset} f \bar{f} d\sigma = -k^2 \operatorname{Re} \int_{\Omega} u \overline{u_0} dx + \operatorname{Re} \int_{\Omega} A^0 \nabla u \cdot \overline{\nabla u_0} dx.$$

Therefore, we have

$$\begin{aligned} \int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma &= \operatorname{Re} \int_{\Omega} (A - A^0) \nabla u \cdot \overline{\nabla u_0} dx \\ &= \operatorname{Re} \int_{\Omega} (\tilde{A} - A^0) \chi_D \nabla u \cdot \overline{\nabla u_0} dx. \end{aligned} \quad (3.9)$$

□

The estimates in the following lemma play an important role in our reconstruction algorithm.

Lemma 3.6. *We have the following identities:*

$$\begin{aligned} \int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma &= - \int_{\Omega} A \nabla w \cdot \overline{\nabla w} dx + k^2 \int_{\Omega} |w|^2 dx \\ &\quad + \int_D (A^0 - \tilde{A}) \nabla u_0 \cdot \overline{\nabla u_0} dx, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma &= \int_{\Omega} A^0 \nabla w \cdot \overline{\nabla w} dx - k^2 \int_{\Omega} |w|^2 dx \\ &\quad + \int_D (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla u} dx. \end{aligned} \quad (3.11)$$

In particular, we have

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma \leq k^2 \int_{\Omega} |w|^2 dx + \hat{\Lambda} \int_D |\nabla u_0|^2 dx, \quad (3.12)$$

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma \geq c \int_{\Omega} |\nabla u_0|^2 dx - k^2 \int_{\Omega} |w|^2 dx, \quad (3.13)$$

where c depending only on $\tilde{\lambda}$ and λ^0 .

Proof. Multiplying the identity

$$\nabla \cdot (A(x) \nabla w) + k^2 w + \nabla \cdot ((\tilde{A} \chi_D - A^0 \chi_D) \nabla u_0) = 0$$

by \bar{w} and integrating over Ω , we get

$$\begin{aligned}
0 &= \int_{\Omega} \nabla \cdot (A \nabla w) \bar{w} dx + \int_{\Omega} \nabla \cdot ((A^0 - \tilde{A}) \chi_D \nabla u_0) \bar{w} dx + k^2 \int_{\Omega} |w|^2 dx \\
&= - \int_{\Omega} A \nabla w \cdot \overline{\nabla w} dx + \int_{\partial\Omega} A \frac{\partial w}{\partial \nu} \bar{w} d\sigma - \int_{\Omega} (A^0 - \tilde{A}) \chi_D \nabla u_0 \cdot \overline{\nabla w} dx \\
&\quad + \int_{\partial\Omega} (A^0 - \tilde{A}) \chi_D \frac{\partial u_0}{\partial \nu} \bar{w} d\sigma + k^2 \int_{\Omega} |w|^2 dx \\
&= - \int_{\Omega} A \nabla w \cdot \overline{\nabla w} dx - \int_D (A^0 - \tilde{A}) \nabla u_0 \cdot \overline{\nabla w} dx + k^2 \int_{\Omega} |w|^2 dx \\
&= \int_{\Omega} A \nabla w \cdot \overline{\nabla w} dx - \int_D (A^0 - \tilde{A}) \nabla u_0 \cdot \overline{\nabla u} dx + k^2 \int_{\Omega} |w|^2 dx \\
&\quad + \int_{\Omega} (A^0 - \tilde{A}) \chi_D \nabla u_0 \cdot \overline{\nabla u_0} dx,
\end{aligned}$$

and use (3.9) we can obtain

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma = - \int_{\Omega} A \nabla w \cdot \overline{\nabla w} dx + \int_D (A^0 - \tilde{A}) \nabla u_0 \cdot \overline{\nabla u_0} dx + k^2 \int_{\Omega} |w|^2 dx.$$

Similarly, multiplying the identity

$$0 = \nabla \cdot ((\tilde{A} - A^0) \chi_D \nabla u) + \nabla \cdot (A^0 \nabla w) + k^2 w = 0$$

by \bar{w} and integrating over Ω , we get

$$\begin{aligned}
0 &= \int_{\Omega} \nabla \cdot ((\tilde{A} - A^0) \chi_D \nabla u) \bar{w} dx + \int_{\Omega} \nabla \cdot (A^0 \nabla w) \bar{w} dx + k^2 \int_{\Omega} |w|^2 dx \\
&= - \int_D (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla w} dx - \int_{\Omega} A^0 \nabla w \cdot \overline{\nabla w} dx + k^2 \int_{\Omega} |w|^2 dx \\
&= - \int_D (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla u} dx - \int_D (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla u_0} dx + k^2 \int_{\Omega} |w|^2 dx \\
&\quad - \int_{\Omega} A^0 \nabla w \cdot \overline{\nabla w} dx,
\end{aligned}$$

and use (3.9) again, we can obtain

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma = \int_{\Omega} A^0 \nabla w \cdot \overline{\nabla w} dx - k^2 \int_{\Omega} |w|^2 dx + \int_D (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla u} dx.$$

For the remaining part, (3.12) is an easy consequence of (3.10)

$$\begin{aligned}
\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f \bar{f} d\sigma &\leq k^2 \int_{\Omega} |w|^2 dx + \int_D (A^0 - \tilde{A}) \nabla u_0 \cdot \overline{\nabla u_0} dx \\
&\leq k^2 \int_{\Omega} |w|^2 dx + \hat{\Lambda} \int_D |\nabla u_0|^2 dx
\end{aligned}$$

Finally, for the lower bound, we use

$$\begin{aligned}
A^0 \nabla w \cdot \overline{\nabla w} + (\tilde{A} - A^0) \nabla u \cdot \overline{\nabla u} &= \tilde{A} \nabla u \cdot \overline{\nabla u} - 2 \operatorname{Re} A^0 \nabla u \cdot \overline{\nabla u_0} + A^0 \nabla u_0 \cdot \overline{\nabla u_0} \\
&= \tilde{A} (\nabla u - (\tilde{A})^{-1} A^0 \nabla u_0) \cdot \overline{(\nabla u - (\tilde{A})^{-1} A^0 \nabla u_0)} \\
&\quad + (A^0 - (\tilde{A})^{-1} (A^0)^2) \nabla u_0 \cdot \overline{\nabla u_0} \\
&\geq (A^0 - (\tilde{A})^{-1} (A^0)^2) \nabla u_0 \cdot \overline{\nabla u_0} \\
&\geq c |\nabla u_0|^2,
\end{aligned}$$

since $\tilde{A}(\nabla u - (\tilde{A})^{-1}A^0\nabla u_0) \cdot \overline{(\nabla u - (\tilde{A})^{-1}A^0\nabla u_0)} \geq 0$ and note that $A^0 - (\tilde{A})^{-1}(A^0)^2 = (\tilde{A})^{-1}(\tilde{A} - A^0)A^0$ has a positive lower bound depending only on $\tilde{\lambda}$ and λ^0 . \square

Before stating our main theorem, we need to estimate $\|w\|_{L^2(\Omega)}$. Fortunately, we can use Meyers L^p estimates to help us to overcome the difficulties (see lemma 3.2 and lemma 3.3). For the upper bound of $\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset)f\bar{f}d\sigma$, see (3.11), we use $\|w\|_{L^2(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$ for $p \leq 2$. Then we have

$$\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset)f\bar{f}d\sigma \leq C\|u_0\|_{W^{1,p}(D)}^2. \quad (3.14)$$

By (3.13) and the Meyers L^p estimate $\|w\|_{L^2(\Omega)} \leq C\|u_0\|_{W^{1,p}(D)}$, we have

$$\int_{\partial\Omega}(\Lambda_D - \Lambda_\emptyset)f\bar{f}d\sigma \geq c \int_{\Omega} |\nabla u_0|^2 dx - c\|u_0\|_{W^{1,p}(D)}^2. \quad (3.15)$$

4 Detecting the convex hull of the unknown obstacle

4.1 Main theorem

Recall that we have constructed the oscillating-decaying solutions in section 2, and note that this solution can not be defined on the whole domain, that is, the oscillating-decaying solutions $u_{\chi_t, b, t, N, \omega}(x, \tau)$ only defined on $\Omega_t(\omega) \subsetneq \Omega$. Nevertheless, with the help of the Runge approximation property, we can prove that one can determine the convex hull of the unknown obstacle D by $\Lambda_D f$ for infinitely many f .

We define B to be an open ball in \mathbb{R}^3 such that $\bar{\Omega} \subset B$. Assume that $\tilde{\Omega} \subset \mathbb{R}^3$ is an open Lipschitz domain with $\bar{B} \subset \tilde{\Omega}$. As in the section 2, set $\omega \in S^2$ and $\{\eta, \zeta, \omega\}$ forms an orthonormal basis of \mathbb{R}^3 . Suppose $t_0 = \inf_{x \in D} x \cdot \omega = x_0 \cdot \omega$, where $x_0 = x_0(\omega) \in \partial D$. For any $t \leq t_0$ and $\epsilon > 0$ small enough, we can construct

$$\begin{aligned} u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} &= \chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x \cdot \xi}e^{-\tau(x \cdot \omega - (t-\epsilon))A_{t-\epsilon}(x')}b + \gamma_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \\ &\quad + r_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \end{aligned}$$

to be the oscillating-decaying solution for $\nabla \cdot (A^0(x)\nabla \cdot) + k^2 \cdot$ in $B_{t-\epsilon}(\omega) = B \cap \{x \cdot \omega > t - \epsilon\}$, where $\chi_{t-\epsilon}(x') \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}$. Note that in section 2, we have assumed the leading coefficient $A^0(x) \in B^\infty(\mathbb{R}^3)$. Similarly, we have the oscillating-decaying solution

$$u_{\chi_t, b, t, N, \omega}(x, \tau) = \chi_t(x')Q_t e^{i\tau x \cdot \xi}e^{-\tau(x \cdot \omega - t)A_t(x')}b + \gamma_{\chi_t, b, t, N, \omega}(x, \tau) + r_{\chi_t, b, t, N, \omega}$$

for L_{A^0} in $B_t(\omega)$. In fact, for any τ , $u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega}(x, \tau) \rightarrow u_{\chi_t, b, t, N, \omega}(x, \tau)$ in an appropriate sense as $\epsilon \rightarrow 0$. For details, we refer readers to consult all the details and results in [2], and we list consequences in the following.

$$\chi_{t-\epsilon}(x')Q_{t-\epsilon}(x')e^{i\tau x \cdot \xi}e^{-\tau(x \cdot \omega - (t-\epsilon))A_{t-\epsilon}(x')}b \rightarrow \chi_t(x')Q_t e^{i\tau x \cdot \xi}e^{-\tau(x \cdot \omega - t)A_t(x')}b$$

in $H^2(B_t(\omega))$ as ϵ tends to 0,

$$\gamma_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \rightarrow \gamma_{\chi_t, b, t, N, \omega}$$

in $H^2(B_t(\omega))$ as ϵ tends to 0, and finally,

$$r_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \rightarrow r_{\chi_t, b, t, N, \omega}$$

in $H^1(B_t(\omega))$ as ϵ tends to 0.

Obviously, $B_{t-\epsilon}(\omega)$ is a convex set and $\overline{\Omega_t(\omega)} \subset B_{t-\epsilon}(\omega)$ for all $t \leq t_0$. By using the Runge approximation property, we can see that there exists a sequence of functions $\tilde{u}_{\epsilon, j}$, $j = 1, 2, \dots$, such that

$$\tilde{u}_{\epsilon, j} \rightarrow u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \text{ in } H^1(B_t(\omega)),$$

where $\tilde{u}_{\epsilon, j} \in H^1(\tilde{\Omega})$ satisfy $L_{A^0} \tilde{u}_{\epsilon, j} = 0$ in $\tilde{\Omega}$ for all ϵ, j . Define the indicator function $I(\tau, \chi_t, b, t, \omega)$ by the formula:

$$I(\tau, \chi_t, b, t, \omega) = \lim_{\epsilon \rightarrow 0} \lim_{j \rightarrow \infty} \int_{\partial} (\Lambda_D - \Lambda_{\emptyset}) f_{\epsilon, j} \overline{f_{\epsilon, j}} d\sigma,$$

where $f_{\epsilon, j} = \tilde{u}_{\epsilon, j}|_{\partial\Omega}$.

Note that in [2], they assume that D satisfying the following condition: For each $\omega \in S^2$, there exist $c_\omega > 0$, $\epsilon_\omega > 0$ and $p_\omega \in [0, 1]$ such that

$$\frac{1}{c_\omega} s^{p_\omega} \leq \mu(\{x \in D | x \cdot \omega = t_0 + s\}) \leq c_\omega s^{p_\omega} \text{ for all } s \in (0, \epsilon_\omega),$$

where μ is the surface measure, but we drop this condition in the following theorem. Now the characterization of the convex hull of D is based on the following theorem:

Theorem 4.1. (1) If $t < t_0$, then for any $\chi_t \in C_0^\infty(\mathbb{R}^2)$ and $b \in \mathbb{C}$, we have

$$\limsup_{\tau \rightarrow \infty} |I(\tau, \chi_t, b, t, \omega)| = 0.$$

(2) If $t = t_0$, then for any $\chi_{t_0} \in C_0^\infty(\mathbb{R}^2)$ with $x'_0 = (x_0 \cdot \eta, x_0 \cdot \zeta)$ being an interior point of $\text{supp}(\chi_{t_0})$ and $0 \neq b \in \mathbb{C}$, we have

$$\liminf_{\tau \rightarrow \infty} |I(\tau, \chi_{t_0}, b, t_0, \omega)| > 0.$$

Proof. (1) Note that we have a sequence of functions $\{\tilde{u}_{\epsilon, j}\}$ satisfies the equation $\nabla \cdot (A^0 \nabla u) + k^2 u = 0$ in Ω , as in the beginning of the section 3, let $w_{\epsilon, j} = u - \tilde{u}_{\epsilon, j}$, then $w_{\epsilon, j}$ satisfies the Dirichlet problem

$$\begin{cases} \nabla \cdot (A(x) \nabla w_{\epsilon, j}) + k^2 w = -\nabla \cdot ((\tilde{A} \chi_D - A^0 \chi_D) \nabla \tilde{u}_{\epsilon, j}) & \text{in } \Omega, \\ w_{\epsilon, j} = 0 & \text{on } \partial\Omega. \end{cases}$$

So we can apply (3.14) directly, which means

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_{\emptyset}) f_{\epsilon, j} \overline{f_{\epsilon, j}} d\sigma \leq C \|\tilde{u}_{\epsilon, j}\|_{W^{1, p}(D)}^2 \leq C \|\tilde{u}_{\epsilon, j}\|_{H^1(D)}^2,$$

where the last inequality obtained by the Hölder's inequality.

By the Runge approximation property we have

$$\tilde{u}_{\epsilon, j} \rightarrow u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega} \text{ in } H^1(B_t(\omega))$$

as $j \rightarrow \infty$ and we know that the obstacle $D \subset B_t(\omega)$, so we have

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega}\|_{H^1(D)} \rightarrow 0$$

as $j \rightarrow \infty$ for all $\epsilon > 0$. Moreover, we know that $u_{\chi_{t-\epsilon},b,t-\epsilon,N,\omega} \rightarrow u_{\chi_t,b,t,N,\omega}$ as $\epsilon \rightarrow 0$ in $H^1(B_t(\omega))$, which implies

$$\|\tilde{u}_{\epsilon,j} - u_{\chi_t,b,t,N,\omega}\|_{H^1(D)} \rightarrow 0$$

as $\epsilon \rightarrow 0, j \rightarrow \infty$. Now by the definition of $I(\tau, \chi_t, b, t, \omega)$, we have

$$I(\tau, \chi_t, b, t, \omega) \leq C \|u_{\chi_t,b,t,N,\omega}\|_{H^1(D)}^2.$$

Now if $t < t_0$, we substitute $u_{\chi_t,b,t,N,\omega} = w_{\chi_t,b,t,N,\omega} + r_{\chi_t,b,t,N,\omega}$ with $w_{\chi_t,b,t,N,\omega}$ being described by (2.3) into

$$I(\tau, \chi_t, b, t, \omega) \leq C \left(\int_D |u_{\chi_t,b,t,N,\omega}|^2 dx + \int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2 dx \right)$$

and use estimates (2.4), (2.5) to obtain that

$$|I(\tau, \chi_t, b, t, \omega)| \leq C \tau^{-2N-1}$$

which finishes

$$\limsup_{\tau \rightarrow \infty} |I(\tau, \chi_t, b, t, \omega)| = 0.$$

For the second part, we use (3.15), which means that we have

$$\int_{\partial\Omega} (\Lambda_D - \Lambda_\emptyset) f_{\epsilon,j} \overline{f_{\epsilon,j}} d\sigma \geq c \int_D |\nabla \tilde{u}_{\epsilon,j}|^2 dx - k^2 \int_\Omega |\tilde{u}_{\epsilon,j}|^2 dx - \int_D |\tilde{u}_{\epsilon,j}|^2 dx.$$

From (4.1) and the similar argument in the first part, it is easy to get

$$I(\tau, \chi_t, b, t, \omega) \geq c \int_D |\nabla u_{\chi_t,b,t,N,\omega}|^2 dx - c \|u_{\chi_t,b,t,N,\omega}\|_{W^{1,p}(D)}^2, \quad (4.1)$$

where $w_{\chi_t,b,t,N,\omega} = u - u_{\chi_t,b,t,N,\omega}$. \square

For the remaining part, we need some extra estimates in the following section.

4.2 End of the proof of Theorem 4.1

In view of the lower bound, we need to introduce the sets $D_{j,\delta} \subset D$, $D_\delta \subset D$ in the following. Recall that $h_D(\omega) = \inf_{x \in D} x \cdot \omega$ and $t_0 = h_D(\omega) = x_0 \cdot \omega$ for some $x_0 \in \partial D$. $\forall \alpha \in \partial D \cap \{x \cdot \rho = h_D(\omega)\} := K$, define $B(\alpha, \delta) = \{x \in \mathbb{R}^3; |x - \alpha| < \delta\}$ ($\delta > 0$). Note $K \subset \cup_{\alpha \in K} B(\alpha, \delta)$ and K is compact, so there exists $\alpha_1, \dots, \alpha_m \in K$ such that $K \subset \cup_{j=1}^m B(\alpha_j, \delta)$. Thus, we define

$$D_{j,\delta} := D \cap B(\alpha_j, \delta) \text{ and } D_\delta := \cup_{j=1}^m D_{j,\delta}.$$

It is easy to see that

$$\int_{D \setminus D_\delta} e^{-p\tau(x \cdot \omega - t_0)A_{t_0}(x')} b dx = O(e^{-pa\tau}),$$

where $A_{t_0}(x') \in B^\infty(\mathbb{R}^2)$ is bounded and its real part strictly greater than 0. so $\exists a > 0$ such that $\operatorname{Re} A_{t_0}(x') \geq a > 0$. Let $\alpha_j \in K$, by rotation and translation, we may assume $\alpha_j = 0$ and the vector $\alpha_j - x_0 = -x_0$ is parallel to $e_3 = (0, 0, 1)$. Therefore, we consider the change of coordinates near each α_j as follows:

$$\begin{cases} y' = x' \\ y_3 = x \cdot \omega - t_0, \end{cases}$$

where $x = (x_1, x_2, x_3) = (x', x_3)$ and $y = (y_1, y_2, y_3) = (y', y_3)$. Denote the parametrization of ∂D near α_j by $l_j(y')$, then we have the following estimates.

Lemma 4.2. *For $q \leq 2$, we have*

$$\begin{aligned} \int_D |u_{\chi_{t_0}, b, t_0, N, \omega}|^q dx &\leq c\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-aq\tau l_j(y')} dy' + O(\tau^{-1} e^{-qa\delta\tau}) \\ &\quad + O(e^{-qa\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1}), \end{aligned} \quad (4.2)$$

$$\begin{aligned} \int_D |u_{\chi_{t_0}, b, t_0, N, \omega}|^2 dx &\geq C\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) \\ &\quad + O(\tau^{-3}) + O(\tau^{-2N-1}), \end{aligned} \quad (4.3)$$

$$\begin{aligned} \int_D |\nabla u_{\chi_{t_0}, b, t_0, N, \omega}|^q dx &\leq C\tau^{q-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-qa\tau l_j(y')} dy' + O(\tau^{-1} e^{-aq\delta\tau}) \\ &\quad + O(e^{-qa\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1}), \end{aligned} \quad (4.4)$$

and

$$\begin{aligned} \int_D |\nabla u_{\chi_{t_0}, b, t_0, N, \omega}|^2 dx &\geq C\tau \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) \\ &\quad + O(\tau^{-1}) + O(\tau^{-2N-1}). \end{aligned} \quad (4.5)$$

Proof. The proof follows from [6]. We only prove (4.2) and (4.3) and the proof of (4.4) and (4.5) are similar arguments.

For (4.2):

$$\begin{aligned}
\int_D |u_{\chi_{t_0}, b, t_0, N, \omega}|^q dx &\leq C \int_D e^{-qa\tau(x \cdot \omega - t_0)} dx + C_q \int_D |\gamma_{\chi_{t_0}, b, t_0, N, \omega}|^q dx \\
&\quad + C_q \int_D |r_{\chi_{t_0}, b, t_0, N, \omega}|^q dx \\
&\leq C \int_{D_\delta} e^{-qa\tau(x \cdot \omega - t_0)} dx + C \int_{D \setminus D_\delta} e^{-qa\tau(x \cdot \omega - t_0)} dx \\
&\quad + C \int_D |\gamma_{\chi_{t_0}, b, t_0, N, \omega}|^2 dx + C \int_D |r_{\chi_{t_0}, b, t_0, N, \omega}|^2 dx \\
&\leq C \sum_{j=1}^m \iint_{|y'| < \delta} dy' \int_{l_j(y')}^\delta e^{-qa\tau y_3} dy_3 + C e^{-qa\tau} \\
&\quad + C \|\gamma_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}^2 + C \|r_{\chi_{t_0}, b, t_0, N, \omega}\|_{H^1(D)}^2 \\
&\leq C \tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-aqa\tau l_j(y')} dy' - \frac{C}{q} \tau^{-1} e^{-qa\delta\tau} \\
&\quad + C e^{-qa\tau} + C \tau^{-3} + C \tau^{-2N-1}
\end{aligned}$$

note that $D \subset \Omega_{t_0}(\omega)$, which proves (4.1).

For (4.3):

$$\begin{aligned}
\int_D |u_{\chi_{t_0}, b, t_0, N, \omega}|^2 dx &\geq C \int_D e^{-2a\tau(x \cdot \omega - t_0)} dx - C \|\gamma_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(\Omega_{t_0}(\omega))}^2 \\
&\quad - C \|r_{\chi_{t_0}, b, t_0, N, \omega}\|_{H^1(\Omega_{t_0}(\omega))}^2 \\
&\geq C \int_{D_\delta} e^{-2a\tau(x \cdot \omega - t_0)} dx - C \tau^{-3} - C \tau^{-2N-1} \\
&= C \tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' - \frac{C}{2} \tau^{-1} e^{-2a\tau} \\
&\quad - C \tau^{-3} - C \tau^{-2N-1}.
\end{aligned}$$

□

Recall that we have (4.1), the lower bound of $I(\tau, \chi_{t_0}, b, t_0, \omega)$, so we want to compare the order (in τ) of $\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}$, $\|\nabla u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}$, $\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^p(D)}$ and $\|\nabla u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^p(D)}$.

Lemma 4.3. For $\max\{2 - \epsilon, \frac{6}{5}\} < p \leq 2$, we have the estimates as follows:

$$\frac{\|\nabla u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}^2}{\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}^2} \geq C \tau^2, \quad \frac{\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^p(\Omega)}^2}{\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}^2} \geq C \tau^{1 - \frac{2}{p}}$$

and

$$\frac{\|\nabla u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^p(D)}^2}{\|u_{\chi_{t_0}, b, t_0, N, \omega}\|_{L^2(D)}^2} \geq C \tau^{3 - \frac{2}{p}}$$

for $\tau \gg 1$.

Proof. The idea of the proof comes from [6], but here we still need to deal with the $\gamma_{\chi_{t_0}, b, t_0, N, \omega}$ and $r_{\chi_{t_0}, b, t_0, N, \omega}$ in $D \subset \Omega_{t_0}(\omega)$. Note that if ∂D is Lipschitz, in our parametrization $l_j(y')$, we have $l_j(y') \leq C|y'|$. Hence,

$$\begin{aligned} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' &\geq C \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2\tau|y'|} dy' \\ &\geq C\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \tau\delta} e^{-2|y'|} dy' \\ &= O(\tau^{-1}). \end{aligned}$$

For simplicity, we define $u_0 := u_{\chi_{t_0}, b, t_0, N, \omega}$ in the following calculations. Using lemma 4.2, we obtain

$$\begin{aligned} &\frac{\int_D |\nabla u_0|^2 dx}{\int_D |u_0|^2 dx} \\ &\geq C \frac{\tau \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) + O(\tau^{-1}) + O(\tau^{-2N-1})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\ &\geq C\tau^2 \frac{1 + \frac{O(\tau^{-2} e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N-2})}{\sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy'}}{1 + \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy'}} \\ &= O(\tau^2) \end{aligned}$$

as $\tau \gg 1$, where

$$\lim_{\tau \rightarrow \infty} \frac{O(\tau^{-2} e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N-2})}{\sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy'} = 0$$

and

$$\lim_{\tau \rightarrow \infty} \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy'} = 0.$$

Now, by using the Hölder's inequality with the exponent $q = \frac{2}{p} \geq 1$, we have

$$\sum_{j=1}^m \iint_{|y'| < \delta} e^{-pa\tau l_j(y')} dy' \leq C \left(\sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' \right)^{\frac{p}{2}}.$$

Hence we use lemma 4.2 again, we have

$$\begin{aligned} &\frac{(\int_D |u_0|^p dx)^{\frac{2}{p}}}{\int_D |u_0|^2 dx} \\ &\leq C \frac{\tau^{-\frac{2}{p}} (\sum_{j=1}^m \iint_{|y'| < \delta} e^{-pa\tau l_j(y')} dy')^{\frac{2}{p}} + O(\tau^{-\frac{2}{p}} e^{-2a\delta\tau}) + O(e^{-2a\tau})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\ &\quad + \frac{O(\tau^{-\frac{6}{p}}) + O(\tau^{-\frac{4N-2}{p}})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \end{aligned}$$

$$\begin{aligned}
&\leq C\tau^{-\frac{2}{p}+1} \frac{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(e^{-2c\delta\tau}) + O(e^{-2a\tau}\tau^{\frac{2}{p}})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})} \\
&\quad + \frac{O(\tau^{-\frac{4}{p}}) + O(\tau^{-\frac{4N}{p}})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\
&= \tau^{-\frac{2}{p}+1} \frac{1 + \frac{O(e^{-2c\delta\tau}) + O(e^{-2c\tau}\tau^{\frac{2}{p}}) + O(\tau^{-\frac{4}{p}}) + O(\tau^{-\frac{4N}{p}})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}}{1 + \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}} \\
&= O(\tau^{-\frac{2}{p}+1})
\end{aligned}$$

as $\tau \gg 1$ and

$$\begin{aligned}
&\frac{(\int_D |\nabla u_0|^p dx)^{\frac{2}{p}}}{\int_D |u_0|^2 dx} \\
&\leq C \frac{\tau^{(p-1)\frac{2}{p}} (\sum_{j=1}^m \iint_{|y'|<\delta} e^{-pa\tau l_j(y')} dy')^{\frac{2}{p}} + O(\tau^{-\frac{2}{p}}e^{-2a\delta\tau}) + O(e^{-2a\tau})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\
&\quad + C \frac{O(\tau^{-\frac{2}{p}}) + O(\tau^{-\frac{4N-2}{p}})}{\tau^{-1} \sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(\tau^{-3}) + O(\tau^{-2N-1})} \\
&\leq C\tau^{3-\frac{2}{p}} \frac{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1}e^{-2a\delta\tau}) + O(e^{-2a\tau}\tau^{\frac{2}{p}-1})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy' + O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})} \\
&\quad + C \frac{O(\tau^{-1}) + O(\tau^{-\frac{4N}{p}-1})}{+O(\tau^{-\frac{2}{p}}) + O(\tau^{-\frac{4N-2}{p}})} \\
&= C\tau^{3-\frac{2}{p}} \frac{1 + \frac{O(\tau^{-1}e^{-2a\delta\tau}) + O(e^{-2a\tau}\tau^{\frac{2}{p}-1}) + O(\tau^{-1}) + O(\tau^{-\frac{4N}{p}-1})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}}{1 + \frac{O(e^{-2a\delta\tau}) + O(\tau^{-2}) + O(\tau^{-2N})}{\sum_{j=1}^m \iint_{|y'|<\delta} e^{-2a\tau l_j(y')} dy'}} \\
&= O(\tau^{3-\frac{2}{p}})
\end{aligned}$$

as $\tau \gg 1$. By (4.1) and above estimates, we have

$$\begin{aligned}
\frac{I(\tau, \chi_t, b, t, \omega)}{\|u_{\chi_t, b, t, N, \omega}\|_{L^2(D)}^2} &\geq C\tau^2 - C\tau^{1-\frac{2}{p}} - C\tau^{3-\frac{2}{p}} \\
&\geq C\tau^2
\end{aligned}$$

for $\tau \gg 1$. On the other hand, for $\|u_{\chi_t, b, t, N, \omega}\|_{L^2(D)}$, we have

$$\begin{aligned}
\int_D |u_{\chi_t, b, t, N, \omega}|^2 dx &\geq C\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau l_j(y')} dy' + O(\tau^{-1} e^{-qa\delta\tau}) \\
&\quad + O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&\geq C\tau^{-1} \sum_{j=1}^m \iint_{|y'| < \delta} e^{-2a\tau |y'|} dy' + O(\tau^{-1} e^{-qa\delta\tau}) \\
&\quad + O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&\geq C\tau^{-2} \sum_{j=1}^m \iint_{|y'| < \tau\delta} e^{-2a|y'|} dy' + O(\tau^{-1} e^{-qa\delta\tau}) \\
&\quad + O(\tau^{-3}) + O(\tau^{-2N-1}) \\
&= O(\tau^{-2}).
\end{aligned}$$

Therefore, we have

$$I(\tau, \chi_t, b, t, \omega) \geq C\tau^2 \|u_{\chi_t, b, t, N, \omega}\|_{L^2(D)}^2 \geq C > 0$$

for $\tau \gg 1$. □

In view of theorem 4.1 and lemma 4.2, we can give an algorithm for reconstructing the convex hull of an inclusion D by the Dirichlet-to-Neumann map Λ_D as long as $A(x)$ and D satisfy the described conditions.

Reconstruction algorithm.

1. Give $\omega \in S^2$ and choose $\eta, \zeta, \xi \in S^2$ so that $\{\eta, \zeta, \xi\}$ forms a basis of \mathbb{R}^3 and ξ lies in the span of η and ζ ;
2. Choose a starting t such that $\Omega \subset \{x \cdot \omega \geq t\}$;
3. Choose a ball B such that the center of B lies on $\{x \cdot \omega = s\}$ for some $s < t$ and $\Omega \subset \overline{B_t(\omega)}$ and take $0 \neq b \in \mathbb{C}$;
4. Choose $\chi_t \in C_0^\infty(\mathbb{R}^2)$ such that $\chi_t > 0$ in $\Sigma_t(\omega)$ and $\chi_t = 0$ on $\partial\Sigma_t(\omega)$;
5. Construct the oscillating-decaying solution $u_{\chi_{t-\epsilon}, b, t-\epsilon, N, \omega}$ in $B_{t-\epsilon}(\omega)$ with $\chi_{t-\epsilon} = \chi_t$ and the approximation sequence $\tilde{u}_{\epsilon, j}$ in $\tilde{\Omega}$;
6. Compute the indicator function $I(\tau, \chi_t, b, t, \omega)$ which is determined by boundary measurements;
7. If $I(\tau, \chi_t, b, t, \omega) \rightarrow 0$ as $\tau \rightarrow \infty$, then choose $t' > t$ and repeat (iv), (v), (vi);
8. If $I(\tau, \chi_t, b, t, \omega) \not\rightarrow 0$ for some $\chi_{t'}$, then $t' = t_0 = h_D(\omega)$;
9. Varying $\omega \in S^2$ and repeat (i) to (viii), we can determine the convex hull of D .

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